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Impulsive Evolution Inclusions with State-Dependent Delay and Multivalued Jumps

Mouffak Benchohra[a1] and **Mohamed Ziane**[b]

[a] Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie
e-mail: benchohra@univ-sba.dz

[b] Département des Mathématiques, Université de Tiaret, B.P. 78, 14000, Tiaret, Algérie
e-mail: ziane.maths@gmail.com

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Abstract

In this paper we prove the existence of a mild solution for a class of impulsive semilinear evolution differential inclusions with state-dependent delay and multivalued jumps in a Banach space. We consider the cases when the multivalued nonlinear term takes convex values as well as nonconvex values.

1 Introduction

In this paper, we are concerned by the existence of mild solution of impulsive semilinear functional differential inclusions with state-dependent delay and multivalued jumps in a Banach space E. More precisely, we consider the following class of semilinear impulsive differential inclusions:

$$x'(t) \in A(t)x(t) + F(t, x_{\rho(t,x_t)}), \quad t \in J = [0, b],\ t \neq t_k, \tag{1.1}$$

$$\Delta x\big|_{t=t_k} \in \mathcal{I}_k(x(t_k^-)), \quad k = 1, \dots, m \tag{1.2}$$

$$x(t) = \phi(t), \quad t \in (-\infty, 0], \tag{1.3}$$

where $\{A(t) : t \in J\}$ is a family of linear operators in Banach space E generating an evolution operator, F be a Carathéodory type multifunction from $J \times \mathcal{B}$ to

[1]Corresponding author

the collection of all nonempty compact convex subsets of E, $\mathcal{B}$ is the phase space defined axiomatically (see section 2) which contains the mapping from $(-\infty, 0]$ into E, $\phi \in \mathcal{B}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\mathcal{I}_k : E \to \mathcal{P}(E)$, $k = 1, \ldots, m$ are multivalued maps with closed, bounded and convex values, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$. Finally $\mathcal{P}(E)$ denotes the family of nonempty subsets of E, $\rho : J \times \mathcal{B} \to (-\infty, b]$.

The theory of impulsive differential equations has become an important area of investigation in recent years, stimulated by the numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, population dynamics, etc. During the last few decades there have been significant developments in impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov [8], Benchohra *et al.* [11], Lakshmikantham *et al.* [29], Samoilenko and Perestyuk [33], and the references therein. For the case where the impulses are absent (i.e. $I_k = 0$, $k = 1, \ldots, m$) and F is a single-valued or multivalued map and A is a densely defined linear operator generating a C_0-semigroup of bounded linear operators and the state space is $C([-r, 0], E)$ or E, the problem (1.1)–(1.3) has been investigated in, for instance, the monographs by Ahmed [4, 5], Hale and Verduyn Lunel [21], Hu and Papageorgiou [26], Kamenskii *et al.* [27] and Wu [34] and the papers by Benchohra and Ntouyas [12], Cardinali and Rubbioni [14], Gory *et al.* [18]. Benedetti [13] considered the existence result in the autonomous case ($A(t) \equiv A$) and finite delay. Cardinali and Rubbioni [15] considered the non autonomous case. In [32] Obukhovskii and Yao considered local and global existence results for semilinear functional differential inclusions with infinite delay and impulse characteristics in a Banach space. Recently some existence results were obtained for certain classes of functional differential equations and inclusions in Banach spaces under assumption that the linear part generates an compact semigroup (see, e.g., [1, 2, 3]).

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received a significant amount of attention in the past several years (we refer to [7, 16, 22, 23, 24] and the references therein). The literature related to functional differential inclusions with state-dependent delay remains limited [1, 3].

Our goal here is to give existence results for the problem (1.1)–(1.3) without any compactness assumption. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we prove existence and compactness of solutions set for problem (1.1)–(1.3). In Section 4, we provide a condition which guarantee the existence of a solution of (1.1)–(1.3) by using a fixed point theorem due to Mönch [31].

We mention that the model with multivalued jump sizes may arise in a control problem where we want to control the jump sizes in order to achieve given objectives. To our knowledge, there are very few results for impulsive evolution inclusions with multivalued jump operators; see [3, 6, 10, 13, 30]. The results of the present paper extend and complement those obtained in the absence of the impulse functions I_k, and those with single-valued impulse functions I_k.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $J := [0, b]$, $b > 0$ and $(E, \|.\|)$ be a real separable Banach space. $C(J, E)$ the space of E-valued continuous functions on J with the uniform norm

$$\|x\|_\infty = \sup\{\|x(t)\|,\ t \in J\}.$$

$L^1(J, E)$ the space of $E-$valued Bochner integrable functions on J with the norm

$$\|f\|_{L^1} = \int_0^b \|f(t)\|dt.$$

To define the solution of problem (1.1)–(1.3), it is convenient to introduce some additional concepts and notations. Consider the following spaces

$$\mathcal{PC}(J, E) = \{y : J \to E, y_k \in C(J_k; E) \text{ there exist } y(t_k^-), y(t_k^+) \text{ with } y(t_k) = y(t_k^-)\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m$. Let the space

$$\mathbf{\Omega} = \left\{y \in (-\infty, b] \to E : \quad y\mid_{(-\infty,0]} \in \mathcal{B} \text{ and } y\mid_J \in \mathcal{PC}(J, E)\right\}$$

with the semi-norm defined by

$$\|y\|_\Omega = \|y_0\|_\mathcal{B} + \sup\{\|y(s)\| : 0 \leq s \leq b\}, \quad y \in \mathcal{PC}.$$

In this work, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [25]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a semi norm $\|.\|_\mathcal{B}$, and satisfies the following axioms introduced at first by Hale and Kato in [20]:

(A1) There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and M locally bounded, such that for any $b > 0$ if $y : (-\infty, b] \to E$, such that $y\mid_J \in \mathcal{PC}(J, E)$ and $y_0 \in \mathcal{B}$; the following conditions hold:

(i) y_t is in $\mathcal{B}$;

(ii) $\|y(t)\| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) $\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{\|y(s)\| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}$ and H, K and M are independent of $y(.)$.

(A2) The space $\mathcal{B}$ is complete.

In what follows we use the following notations $K_b = \sup\{K(t), t \in J\}$ and $M_b = \sup\{M(t), t \in J\}$.

Definition 2.1. Let X and Y be two topological vector spaces. We denote by $\mathcal{P}(Y)$ the family of all non-empty subsets of Y and by

$$\mathcal{P}_k(Y) = \{C \in \mathcal{P}(Y) : \text{compact}\},\ \mathcal{P}_b(Y) = \{C \in \mathcal{P}(Y) : \text{bounded}\},$$
$$\mathcal{P}_c(Y) = \{C \in \mathcal{P}(Y) : \text{closed}\},\ \mathcal{P}_{cv}(Y) = \{C \in \mathcal{P}(Y) : \text{convex}\}.$$

A multifunction $G : X \to \mathcal{P}(Y)$ is said to be upper semicontinuous (u.s.c.) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. The multifunction G is called closed if its graph $\Gamma_G = \{(x, y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. The multifunction G is called quasicompact restriction to any compact subset $M \subset X$ is compact. A multifunction $\mathcal{F} : [c, d] \subset \mathbb{R} \to \mathcal{P}_k(Y)$ is said to be strongly measurable if there exists a sequence $\mathcal{F}_n : [c, d] \to \mathcal{P}_k(Y)$, $n = 1, 2, \ldots$ of steps multifunctions such that

$$\lim_{n\to+\infty} h(\mathcal{F}_n(t), \mathcal{F}(t)) = 0, \quad \text{for } \mu\text{-a.e } t \in [c, d],$$

where μ denotes the Lebesgue measure on $[c, d]$ and h is the Hausdorff metric on $\mathcal{P}_k(Y)$.

A subset B of $L^1([0, b]; E)$ is decomposable if for all $u(.); v(.) \in B$ and $I \subset [0, b]$ measurable, the function $u(.)\mathcal{X}_I + v(.)\mathcal{X}_{[0,b]} \in B$, where $\mathcal{X}$ denotes the characteristic function.

Definition 2.2. Let $F : [0, b] \to \mathcal{P}(E)$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator

$$\mathcal{F} : J \times \mathcal{B} \to \mathcal{P}(L^1([0; b]; E)),$$

defined by

$$\mathcal{F}(x(.)) = \{y(.) \in L^1([0, b]; E) : y(t) \in F(t; x_{\rho(t,x_t)}), \quad \text{for a.e. } t \in [0, b]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say $\mathcal{F}$ is the lower semi-continuous type if its associated Niemytzki operator F is lower semi-continuous and has nonempty closed and decomposable values. For details and equivalent definitions see [19, 27, 28].

Let us recall the following result that will be used in the sequel.

Lemma 2.3. *[9] Let E be a separable metric space and let $G : E \to \mathcal{P}(L^1([0,b];E))$ be a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then G has a continuous selection, i.e. there exists a continuous function $f : E \to L^1([0,b];E)$ such that $f(y) \in G(y)$ for every $y \in E$.*

Definition 2.4. Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : \mathcal{P}_b(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{P}_b(E)$.

Definition 2.5. A measure of noncompactness β is called:

(i) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}_b(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in \mathcal{P}_b(E)$;

(iii) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As an example of the measure of noncompactness possessing all these properties is the Hausdorff of MNC which is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net}\}.$$

For more information about the measure of noncompactness we refer the reader to [27].

Definition 2.6. A multifunction $G : E \to \mathcal{P}_k(E)$ is said to be χ-condensing if for every bounded subset $\Omega \subseteq E$ the relation

$$\chi(G(\Omega)) \geq \chi(\Omega)$$

implies the relative compactness of Ω.

Definition 2.7. A countable set $\{f_n : n \geq 1\} \subseteq L^1(J, E)$ is said to be semicompact if

(i) it is integrably bounded: $\|f_n(t)\| \leq \omega(t)$ for a.e. $t \in J$ and every $n \geq 1$ where $\omega \in L^1(J, \mathbb{R}^+)$

(ii) the set $\{f_n(t) : n \geq 1\}$ is relatively compact in E for a.e. $t \in J$.

Now, let for every $t \in J$, $A(t) : E \to E$ be a linear operator such that

(i) For all $t \in J$, $D(A(t)) = D(A) \subseteq E$ is dense and independent of t.

(ii) For each $s \in I$ and each $x \in E$ there is a unique solution $v : [s, b] \to E$ for the evolution equation

$$\begin{aligned} v'(t) &= A(t)v(t), \quad t \in [s, b] \\ v(s) &= x. \end{aligned} \tag{2.1}$$

In this case an operator T can be defined as

$$T : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E), \quad T(t, s)(x) = v(t),$$

where v is the unique solution of (2.1) and $\mathcal{L}(E)$ is the family of linear bounded operators on E.

Definition 2.8. The operator T is called the evolution operator generated by the family $\{A(t) : t \in J\}$.

1. $T(s, s) = I_E$,
2. $T(t, r)T(r, s) = T(t, s)$, for all $0 \leq s \leq r \leq t \leq b$.
3. $(t, s) \to T(t, s)$ is strongly continuous on Δ and

$$\frac{\partial T(t, s)}{\partial t} = A(t)T(t, s), \quad \frac{\partial T(t, s)}{\partial s} = -T(t, s)A(s).$$

Definition 2.9. The operator $G : L^1(J, E) \to C(J, E)$ defined by

$$Gf(t) = \int_0^t T(t, s)f(s)ds \tag{2.2}$$

is called the generalized Cauchy operator, where $T(.,.)$ is the evolution operator generated by the family of operators $\{A(t) : t \in J\}$.

In the sequel we will need the following results.

Lemma 2.10. *[27] Every semicompact set in $L^1(J, E)$ is weakly compact in the space $L^1(J, E)$.*

Lemma 2.11 ([27, Theorem 2])**.** *The generalized Cauchy operator G satisfies the properties*

(G1) there exists $\zeta \geq 0$ such that

$$\|Gf(t) - Gg(t)\| \leq \zeta \int_0^t \|f(s) - g(s)\|ds, \text{ for every } f, g \in L^1(J, E), \ t \in J.$$

(G2) for any compact $K \subseteq E$ and sequence $(f_n)_{n\geq 1}$, $f_n \in L^1(J,E)$ such that for all $n \geq 1$, $f_n(t) \in K$, a. e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ in $L^1(J,E)$ implies the convergence $Gf_n \to Gf_0$ in $C(J,E)$.

Lemma 2.12. *[27] Let $S : L^1(J,E) \to C(J,E)$ be an operator satisfying condition (G2) and the following Lipschitz condition (weaker than (G1)).*

(G1')

$$\|Sf - Sg\|_{C(J,E)} \leq \zeta \|f - g\|_{L^1(J,E)}.$$

Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(J,E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J,E)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to f_0 in $L^1(J,E)$ then $Sf_n \to Sf_0$ in $C(J,E)$.

Lemma 2.13. *[27] Let $S : L^1(J,E) \to C(J,E)$ be an operator satisfying conditions* (G1), (G2) *and let the set $\{f_n\}_{n=1}^{\infty}$ be integrably bounded with the property $\chi(\{f_n(t) : n \geq 1\}) \leq \eta(t)$, for a.e. $t \in J$, where $\eta(.) \in L^1(J,\mathbb{R}^+)$ and χ is the Hausdorff MNC. Then*

$$\chi(\{Sf_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s)ds, \text{ for all } t \in J,$$

where $\zeta \geq 0$ is the constant in condition (G1).

Lemma 2.14. *[27] If U is a closed convex subset of a Banach space E and $R : U \to \mathcal{P}_{cv,k}(E)$ is a closed β-condensing multifunction, where β is a nonsingular MNC defined on the subsets of U. Then R has a fixed point.*

Lemma 2.15. *[27] Let W be a closed subset of a Banach space E and $R : W \to P_{cv,k}(E)$ be a closed multifunction which is β-condensing on every bounded subset of W, where β is a monotone measure of noncompactness. If the fixed points set $\mathcal{F}ixR$ is bounded, then it is compact.*

Theorem 2.16. *[31] Let E be a Banach space, U an open subset of E and $0 \in U$. Suppose that $N : U \to E$ is a continuous map which satisfies Mönch's condition (that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{co}(\{0\} \cup N(D))$, then $\overline{D}$ is compact) and assume that*

$$x \neq \lambda N(x), \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1)$$

holds. Then Nhas a fixed point in $\overline{U}$.

3 Existence Theorem

In this section we prove the existence of mild solutions for the impulsive semilinear functional differential inclusions (1.1)–(1.3). We will always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. In addition, we introduce the following hypotheses.

(A) $\{A(t) : t \in J\}$ be a family of linear (not necessarily bounded) operators, $A(t) : D(A) \subset E \to E$, $D(A)$ not depending on t and dense subset of E and $T : \Delta = \{(t,s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(E)$ be the evolution operator generated by the family $\{A(t) : t \in J\}$.

(Hϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

(H1) The multifunction $F(.,x)$ has a strongly measurable selection for every $x \in \mathcal{B}$.

(H2) The multifunction $F : (t,.) \to P_{cv,k}(E)$ is upper semicontinuous for a.e. $t \in J$.

(H3) there exists a function $\alpha \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,\psi)\| \leq \alpha(t)(1+\|\psi\|_{\mathcal{B}}) \quad \text{for a.e. } t \in J;$$

(H4) There exists a function $\beta \in L^1(J,\mathbb{R}^+)$ such that for all $\Omega \subset \mathcal{B}$, we have

$$\chi(F(t,\Omega)) \leq \beta(t) \sup_{-\infty \leq s \leq 0} \chi(\Omega(s)) \quad \text{for a.e. } t \in J,$$

where, $\Omega(s) = \{x(s); x \in \Omega\}$ and χ is the Hausdorff measure of noncompactness.

(H5) There exist constants $a_k, c_k > 0, k = 1,\dots,m$ such that

1) $\|I_k\| \leq a_k, \quad$ where $I_k \in \mathcal{I}_k(x(t_k^+))$.

2) $\chi(I_k(D)) \leq c_k \chi(D)$ for each bounded subset D of E.

The next result is a consequence of the phase space axioms.

Lemma 3.1. ([22], Lemma 2.1) *If $y : (-\infty, b] \to \mathbb{R}$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J,\mathbb{R})$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{\|y(\theta)\|; \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where

$$L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t).$$

Remark 3.2. *We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [25] then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L \sup\{\|\phi(\theta)\| : \theta \in (-\infty, 0]\}$ for every $\phi \in \mathcal{B}$ that is continuous and bounded (see [25] Proposition 7.1.1) for details. Consequently,*

$$\|\phi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}} \|\phi\|_{\mathcal{B}}, \quad \text{for every } \phi \in \mathcal{B} \setminus \{0\}.$$

Definition 3.3. *A function $x \in \mathbf{\Omega}$ is said to be a mild solution of system* (1.1)–(1.3) *if there exist a function $f \in L^1(J; E)$ such that $f \in F(t, x_{\rho(t,x_t)})$ for a.e. $t \in J$*

(i) $x(t) = T(t, 0)\phi(0) + \int_0^t T(t, s)f(s)ds + \sum_{0<t_k<t} T(t, t_k)I_k(x(t_k))$, *a.e.* $t \in J,\ k = 1, \ldots, m$

(ii) $x(t) = \phi(t), \quad t \in (-\infty, 0]$,

with $I_k \in \mathcal{I}_k(x(t_k^+))$.

Remark 3.4. *Under conditions (Hϕ) and (H1)-(H3) for every piecewise continuous function $v : J \to \mathcal{B}$ the multifunction $F(t, v(t))$ admits a Bochner integrable selection (see [27]).*

Let

$$\mathbf{\Omega}_b = \{x \in \mathbf{\Omega} : \ x_0 = 0\}.$$

For any $x \in \mathbf{\Omega}_b$ we have

$$\|x\|_b = \|x\|_{\mathcal{B}} + \sup_{0 \leq s \leq b} \|x\| = \sup_{0 \leq s \leq b} \|x\|.$$

Thus $(\mathbf{\Omega}_b, \|.\|_b)$ is a Banach space.

We note that from assumptions (H1) and (H3) it follows that the superposition multioperator $S_F^1 : \mathbf{\Omega}_b \to \mathcal{P}(L^1(J, E))$ defined by

$$S_F^1 = \{f \in L^1(J, E) : f(t) \in F(t, x_{\rho(t,x_t)}), \quad \text{a.e. } t \in J\}$$

is nonempty set (see [27]) and is weakly closed in the following sense.

Lemma 3.5. *If we consider the sequence $(x^n) \in \mathbf{\Omega}_b$ and $\{f_n\}_{n=1}^{+\infty} \subset L^1(J, E)$, where $f_n \in S^1_{F(.,x^n_{\rho(.,x^n_.)})}$ such that $x^n \to x^0$ and $f_n \to f^0$ then $f^0 \in S_F^1$.*

Now we state and prove our main result.

Theorem 3.6. *Under assumptions* (A)–(Hϕ) *and* (H1)–(H5)*, the problem* (1.1)–(1.3) *has at least one mild solution.*

Proof. To prove the existence of a mild solution for (1.1)–(1.3) we introduce the integral multioperator $N : \mathbf{\Omega}_b \longrightarrow \mathcal{P}(\mathbf{\Omega}_b)$, defined as

$$Nx = \begin{cases} y: \ y(t) = T(t,0)\phi(0) + \int_0^t T(t,s)f(s)ds \\ \quad \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k)), & t \in J \\ y(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{3.1}$$

where S_F^1 and $I_k \in \mathcal{I}_k(x)$.

It is clear that the integral multioperator N is well defined and the set of all mild solution for the problem (1.1)–(1.3) on J is the set $\mathcal{F}ixN = \{x : x \in N(x)\}$.

We shall prove that the integral multioperator N satisfies all the hypotheses of Lemma 2.14. The proof will be given in several steps.

Step 1. Using the fact that the maps F and $\mathcal{I}$ has a convex values it easy to check that N has convex values.

Step 2. N has closed graph.

Let $\{x^n\}_{n=1}^{+\infty} \subset \mathbf{\Omega}_b$, $\{z^n\}_{n=1}^{+\infty}$, $x^n \to x^*$, $z^n \in N((x^n), n \geq 1)$ and $z^n \to z^*$. Moreover, let $\{f_n\}_{n=1}^{+\infty} \subset L^1(J;E)$ an arbitrary sequence such that $f_n \in S_F^1$ for $n \geq 1$.

Hypothesis (H3) implies that the set $\{f_n\}_{n=1}^{+\infty}$ integrably bounded and for a.e. $t \in J$ the set $\{f_n(t)\}_{n=1}^{+\infty}$ relatively compact, we can say that $\{f_n\}_{n=1}^{+\infty}$ is semicompact sequence. Consequently $\{f_n\}_{n=1}^{+\infty}$ is weakly compact in $L^1(J;E)$, so we can assume that $f_n \rightharpoonup f^*$.

From lemma 2.11 we know that the generalized Cauchy operator on the interval J, $G : L^1(J;E) \to \mathbf{\Omega}_b$, defined by

$$Gf(t) = \int_0^t T(t,s)f(s)ds, \quad t \in J \tag{3.2}$$

satisfies properties (G1) and (G2) on J.

Note that set $\{f_n\}_{n=1}^{+\infty}$ is also semicompact and sequence $(f_n)_{n=1}^{+\infty}$ weakly converges to f^* in $L^1(J;E)$. Therefore, by applying Lemma 2.12 for the generalized Cauchy operator G of (3.2) we have the convergence $Gf_n \to Gf$. By means of

(3.2) and (3.1), for all $t \in J$ we can write

$$\begin{aligned} z_n(t) &= T(t,0)\phi(0) + \int_0^t T(t,s)f_n(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k)) \\ &= T(t,0)\phi(0) + \int_0^t T(t,s)f_n ds + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k)) \\ &= T(t,0)\phi(0) + Gf_n(t) + \sum_{0<t_k<t} T(t,t_k)I_k(x^n(t_k))) \end{aligned}$$

where S^1_F, and $I_k \in \mathcal{I}_k(x)$.

By applying Lemma 2.11, we deduce

$$z_n \to T(.,0)\phi(0) + Gf + T(.,t)I_k(x^*(t_k))$$

in $\mathbf{\Omega}_b$ and by using in fact that the operator S^1_F is closed, we get $f^* \in S^1_F$. Consequently

$$z^*(t) \to T(t,0)\phi(0) + Gf + T(t,t)I_k(x^*(t_k)),$$

therefore $z^* \in N(x^*)$. Hence N is closed.

With the same technique, we obtain that N has compact values.

Step 3. We consider the measure of noncompactness defined in the following way. For every bounded subset $\Omega \subset \mathbf{\Omega}_b$

$$\nu_1(\Omega) = \max_{\Omega\in\Delta(\Omega)} (\gamma_1(\Omega), \quad \operatorname{mod}_C(\Omega)), \tag{3.3}$$

where $\Delta(\Omega)$ is the collection of all the denumerable subsets of Ω;

$$\gamma_1(\Omega) = \sup_{t\in J} e^{-Lt}\chi(\{x(t) : x \in \Omega\}); \tag{3.4}$$

where $\operatorname{mod}_C(\Omega)$ is the modulus of equicontinuity of the set of functions Ω given by the formula

$$\operatorname{mod}_C(\Omega) = \lim_{\delta\to 0}\sup_{x\in\Omega}\max_{|t_1-t_2|\le\delta} \|x(t_1) - x(t_2)\|; \tag{3.5}$$

and $L > 0$ is a positive real number chosen such that

$$q := M\left(2\sup_{t\in J}\int_0^t e^{-L(t-s)}\beta(s)ds + e^{Lt}\sum_{k=1}^m c_k\right) < 1 \tag{3.6}$$

where $M = \sup_{(t,s)\in\Delta} \|T(t,s)\|$.

From the Arzela-Ascoli theorem, the measure ν_1 give a nonsingular and regular measure of noncompactness, (see [27]).

Let $\{y_n\}_{n=1}^{+\infty}$ be the denumerable set which achieves that maximum $\nu_1(N(\Omega))$, i,e;

$$\nu_1(N(\Omega)) = (\gamma_1(\{y_n\}_{n=1}^{+\infty}), \quad \bmod_C(\{y_n\}_{n=1}^{+\infty})).$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n \in N(x_n)$, $n \geq 1$. Then

$$y_n(t) = T(t,0)\phi(0) + \int_0^t T(t,s)f(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k)), \tag{3.7}$$

where $f \in S_F^1$ and $I_k \in \mathcal{I}_k(x_n)$, so that

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{Gf_n\}_{n=1}^{+\infty}).$$

We give an upper estimate for $\gamma_1(\{y_n\}_{n=1}^{+\infty})$.

Fixed $t \in J$ by using condition (H4), for all $s \in [0,t]$ we have

$$\begin{aligned}\chi(\{f_n(s)\}_{n=1}^{+\infty}) &\leq \chi(F(s,\{x_n(s)\}_{n=1}^{+\infty}))\\ &\leq \chi(\{F(s,x_n(s))\}_{n=1}^{+\infty})\\ &\leq \beta(s)\chi(\{x_n(s)\}_{n=1}^{+\infty})\\ &\leq \beta(s)e^{Ls}\sup_{t\in J} e^{-Lt}\chi(\{x_n(t)\}_{n=1}^{+\infty})\\ &= \beta(s)e^{Ls}\gamma_1(\{x_n\}_{n=1}^{+\infty}).\end{aligned}$$

By using condition (H3), the set $\{f_n\}_{n=1}^{+\infty}$ is integrably bounded. In fact, for every $t \in J$, we have

$$\begin{aligned}\|f_n(t)\| &\leq \|F(t,x_n(t))\|\\ &\leq \alpha(t)(1+\|x_n(t\|).\end{aligned}$$

The integrably boundedness of $\{f_n\}_{n=1}^{+\infty}$ follows from the continuity of x in J_k and the boundedness of set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$. By applying Lemma 2.13, it follows that

$$\begin{aligned}\chi(\{Gf_n(s)\}_{n=1}^{+\infty}) &\leq 2M\int_0^s \beta(t)e^{Lt}(\gamma_1(\{x_n\}_{n=1}^{+\infty}))dt\\ &= 2M\gamma_1(\{x_n\}_{n=1}^{+\infty})\int_0^s \beta(t)e^{Lt}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\gamma_1(\{x_n\}_{n=1}^{+\infty}) &\leq \gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{Gf_n(s)\}_{n=1}^{+\infty})\\ &= \sup_{t\in J} e^{-Lt}2M\gamma_1(\{x_n\}_{n=1}^{+\infty})\int_0^s \beta(t)e^{Lt}M\gamma_1(\{x_n\}_{n=1}^{+\infty})e^{Lt}\sum_{k=1}^m c_k \qquad (3.8)\\ &\leq q\gamma_1(\{x_n\}_{n=1}^{+\infty}),\end{aligned}$$

and hence $\gamma_1(\{x_n\}_{n=1}^{+\infty}) = 0$, then $\gamma_1(\{x_n(t)\}_{n=1}^{+\infty}) = 0$, for every $t \in J$. Consequently

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = 0.$$

By using the last equality and hypotheses (H3) and (H4) we can prove that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact. Now, by applying Lemma 2.11 and Lemma 2.12, we can conclude that set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact. The representation of y_n given by (3.7) yields that set $\{y_n\}_{n=1}^{+\infty}$ is also relatively compact in $\mathbf{\Omega}_b$, therefore $\nu_1(\Omega) = (0,0)$. Then Ω is a relatively compact set.

Step 4. A priori bounds.

We will demonstrate that the solution set is a priori bounded. Indeed, let $x \in N$. Then there exists $f \in S_F^1$ and $I_k \in \mathcal{I}_k(x)$ such that for every $t \in J$ we have

$$\begin{aligned}\|x(t)\| &= \Big\|T(t,0)\phi(0) + \int_0^t T(t,s)f(s)ds + \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k))\Big\| \\ &\leq M(\|\phi(0)\| + \sum_{k=1}^m a_k) + M\int_0^t f(s)ds \\ &\leq M(\|\phi(0)\| + \sum_{k=1}^m a_k) + M\int_0^b \alpha(s)(1 + \|x[\phi]_{\rho(t,x_t)}\|)ds.\end{aligned}$$

Using Lemma 3.1, we have

$$\begin{aligned}\|x(t)\| &\leq M(\|\phi(0)\| + \sum_{k=1}^m a_k) + M\int_0^t \alpha(s)(1 + (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup_{0\leq\theta\leq s}\|x(\theta)\|)ds \\ &\leq M(\|\phi(0)\| + \sum_{k=1}^m a_k) + M(1 + (M_b + L^\phi)\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)} \\ &\quad + MK_b\int_0^t \alpha(s)\sup_{0\leq\theta\leq s}\|x(\theta)\|ds.\end{aligned}$$

Since the last expression is a nondecreasing function of t, we have that

$$\begin{aligned}\sup_{0\leq\theta\leq t}\|x(\theta)\| \leq{}& M(\|\phi(0)\| + \sum_{k=1}^m a_k) + M(1 + (M_b + L^\phi)\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)} \\ &+ MK_b\int_0^t \alpha(s)\sup_{0\leq\theta\leq s}\|x(\theta)\|ds.\end{aligned}$$

Invoking Gronwall's inequality, we get

$$\sup_{0\leq\theta\leq b}\|x(\theta)\| \leq \zeta e^{MK_b\|\alpha\|_{L^1[0,b]}},$$

where

$$\zeta = M(\|\phi(0)\| + \sum_{k=1}^{m} a_k) + M(1 + (M_b + L^{\phi})\|\phi\|_{\mathcal{B}})\|\alpha\|_{L^1(J)},$$

which completes the proof. □

4 The nonconvex case

This section is devoted to proving the existence of solutions for (1.1)–(1.3) with a nonconvex valued right-hand side. Our result is based on Mönch's fixed point theorem combined with a selection theorem due to Bressan and Colombo (see [9]). We will assume the following hypothesis:

Let F be a multifunction defined from $J \times \mathcal{B}$ to the family of nonempty closed convex subsets of E such that

(H6) $(t, x) \mapsto F(., x)$ is $\mathcal{L} \otimes \mathcal{B}_b$-measurable ($\mathcal{B}_b$ is Borel measurable).

(H7) The multifunction $F : (t, .) \to P_k(E)$ is lower semicontinuous for a.e. $t \in J$.

(H8) there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, \psi)\| \leq \alpha(t), \quad \text{for a.e. } t \in J, \ \forall \psi \in \mathcal{B};$$

(H9) There exists a function $\beta \in L^1(J, \mathbb{R}^+)$ such that for all $\Omega \subset \mathcal{B}$, we have

$$\chi(F(t, \Omega)) \leq \beta(t) \sup_{-\infty \leq s \leq 0} \chi(\Omega(s)) \quad \text{for a.e. } t \in J,$$

where, $\Omega(s) = \{x(s); x \in \Omega\}$ and χ is the Hausdorff measure of noncompactness.

(H10) There exist constants $a_k, b_k, c_k \geq 0$, $k = 1, \dots, m$, such that

1) $\|I_k\| \leq a_k\|x\| + b_k, \quad$ where $I_k \in \mathcal{I}_k(x(t_k^+))$.

2) $\chi(I_k(D)) \leq c_k \chi(D)$ for each bounded subset D of E.

Now we state and prove our main result.

Theorem 4.1. *Assume that* (A)–(Hϕ) *and* (H6)–(H10) *hold. If*

$$M \sum_{k=1}^{m} a_k < 1,$$

then the problem (1.1)–(1.3) *has at least one mild solution.*

Proof. We note that from assumptions (H6) and (H8) it follows that the superposition multioperator

$$S_F^1 : \mathbf{\Omega}_b \to \mathcal{P}(L^1(J,E)),$$

defined by

$$S_F^1 = \{f \in L^1(J,E) : f(t) \in F(t, x_{\rho(t,x_t)}), \quad \text{a.e. } t \in J\}$$

is nonempty set (see [27]).

Step 1. The Mönch's condition holds.

Suppose that $\Omega \subseteq B_r$ is countable and $\Omega \subseteq \overline{co}(\{0\} \cup N(\Omega))$ We will prove that Ω is relatively compact. We consider the measure of noncompactness defined in (3.3) and $L > 0$ is a positive real number chosen such that

$$q := M \left(2 \sup_{t \in J} \int_0^t e^{-L(t-s)} \beta(s) ds + e^{Lt} \sum_{k=1}^{m} c_k \right) < 1 \tag{4.1}$$

where $M = \sup_{(t,s) \in \Delta} \|T(t,s)\|$.

From the Arzela-Ascoli theorem, the measure ν_1 give a nonsingular and regular measure of noncompactness, (see [27]).

Let $\{y_n\}_{n=1}^{+\infty}$ be the denumerable set which achieves that maximum $\nu_1(N(\Omega))$, i,e;

$$\nu_1(N(\Omega)) = (\gamma_1(\{y_n\}_{n=1}^{+\infty}), \quad \bmod{}_C(\{y_n\}_{n=1}^{+\infty})).$$

Then there exists a set $\{x_n\}_{n=1}^{+\infty} \subset \Omega$ such that $y_n \in N(x_n)$, $n \geq 1$. Then

$$y_n(t) = T(t,0)\phi(0) + \int_0^t T(t,s) f(s) ds + \sum_{0 < t_k < t} T(t,t_k) I_k, \tag{4.2}$$

where $f \in S_F^1$ and $I_k \in \mathcal{I}_k(x_n)$, so that

$$\gamma_1(\{y_n\}_{n=1}^{+\infty}) = \gamma_1(\{G f_n\}_{n=1}^{+\infty}).$$

We give an upper estimate for $\gamma_1(\{y_n\}_{n=1}^{+\infty})$.

Fixed $t \in J$ by using condition (H9), for all $s \in [0,t]$ we have

$$\begin{aligned}
\chi(\{f_n(s)\}_{n=1}^{+\infty}) &\leq \chi(F(s, \{x_n(s)\}_{n=1}^{+\infty})) \\
&\leq \beta(s) \chi(\{x_n(s)\}_{n=1}^{+\infty}) \\
&\leq \beta(s) e^{Ls} \sup_{t \in J} e^{-Lt} \chi(\{x_n(t)\}_{n=1}^{+\infty}) \\
&= \beta(s) e^{Ls} \gamma_1(\{x_n\}_{n=1}^{+\infty}).
\end{aligned}$$

By using condition (H8), the set $\{f_n\}_{n=1}^{+\infty}$ is integrably bounded. In fact, for every $t \in J$, we have

$$\begin{aligned}\|f_n(t)\| &\leq \|F(t, x_n(t))\| \\ &\leq \alpha(t).\end{aligned}$$

By applying Lemma 2.13, it follows that

$$\begin{aligned}\chi(\{Gf_n(s)\}_{n=1}^{+\infty}) &\leq 2M \int_0^s \beta(t) e^{Lt} (\gamma_1(\{x_n\}_{n=1}^{+\infty})) dt \\ &= 2M\gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_0^s \beta(t) e^{Lt} dt.\end{aligned}$$

Thus, we get

$$\begin{aligned}\gamma_1(\{x_n\}_{n=1}^{+\infty}) &\leq \gamma_1(\{y_n\}_{n=1}^{+\infty}) \\ &= \sup_{t\in J} e^{-Lt} 2M\gamma_1(\{x_n\}_{n=1}^{+\infty}) \int_0^t \beta(s) e^{Ls} ds + M\gamma_1(\{x_n\}_{n=1}^{+\infty}) e^{Lt} \sum_{k=1}^{m} c_k \\ &\leq q\gamma_1(\{x_n\}_{n=1}^{+\infty}),\end{aligned} \tag{4.3}$$

Therefore, we have that

$$\gamma_1(\{x_n\}_{n=1}^{+\infty}) \leq \gamma_1(\Omega) \leq \gamma_1(\{0\} \cup N(\Omega))\gamma_1(\{y_n\}_{n=1}^{+\infty}) \leq q\gamma_1(\{x_n\}_{n=1}^{+\infty}).$$

From (3.6), we obtain that

$$\gamma_1(\{x_n\}_{n=1}^{+\infty}) = \gamma_1(\Omega) = \gamma_1(\{y_n\}_{n=1}^{+\infty})$$

Coming back to the definition of γ_1, we can see

$$\chi(\{x_n\}_{n=1}^{+\infty}) = \chi(\{y_n\}_{n=1}^{+\infty}) = 0$$

By using the last equality and hypotheses (H8) and (H9) we can prove that set $\{f_n\}_{n=1}^{+\infty}$ is semicompact. Now, by applying Lemma 2.11 and Lemma 2.12, we can conclude that set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact.

The representation of y_n given by (4.2) yields that set $\{y_n\}_{n=1}^{+\infty}$ is also relatively compact in $\mathbf{\Omega}_b$, since ν_1 is a monotone, nonsingular, regular MNC, we have that

$$\nu_1(\Omega) \leq \nu_1(\overline{co}(\{0\} \cup N(\Omega))) \leq \nu_1(N(\Omega)) = \nu_1(\{y_n\}_{n=1}^{+\infty}) = (0, 0).$$

Therefore, Ω is relatively compact.

Step 2. It is clear that the superposition multioperator S_F^1 has closed and decomposable values. Following the lines of [27], we may verify that S_F^1 is l.s.c..

Applying Lemma 2.3 to the restriction of S_F^1 on $\mathbf{\Omega}_b$ we obtain that there exists a continuous selection

$$w : \mathbf{\Omega}_b \to L^1(J, E)$$

We consider a map $N : \mathbf{\Omega}_b \to \mathbf{\Omega}_b$ defined as

$$x(t) = T(t,0)\phi(0) + \int_0^t T(t,s)w(x)(s)ds$$

Since the Cauchy operator is continuous, the map N is also continuous, therefore, it is a continuous selection of the integral multioperator.

Step 3. A priori bounds.

We will demonstrate that the solution set is a priori bounded. Indeed, let $x \in \lambda N_1$ and $\lambda \in (0,1)$. There exists $f \in S_F^1$ and $I_k \in \mathcal{I}_k(x)$ such that for every $t \in J$ we have

$$\begin{aligned}\|x(t)\| &= \Big\|\lambda T(t,0)\phi(0) + \lambda\int_0^t T(t,s)f(s)ds + \lambda \sum_{0<t_k<t} T(t,t_k)I_k\Big\|, \\ &\leq M(\|\phi(0)\| + \|x\|\sum_{k=1}^m a_k + \sum_{k=1}^m b_k) + M\int_0^t \alpha(s)ds,\end{aligned}$$

hence,

$$(1 - M\sum_{k=1}^m a_k)\|x\| \leq M(\|\phi(0)\| + \|\alpha\|_{L^1} + \sum_{k=1}^m b_k).$$

Consequently

$$\|x\| \leq \frac{M(\|\phi(0)\| + \|\alpha\|_{L^1} + \sum_{k=1}^m b_k)}{1 - M\sum_{k=1}^m a_k} = C.$$

So, there exists N^* such that $\|x\| \neq N^*$, set

$$U = \{x \in \mathbf{\Omega}_b : \quad \|x\| < N^*\}.$$

From the choice of U there is no $x \in \partial U$ such that $x = \lambda N x$ for some $\lambda \in (0,1)$.

Thus, we get a fixed point of N_1 in $\bar{U}$ due to the Mönch Theorem. □

5 An example

As an application of our results we consider the following impulsive partial functional differential equation of the form

$$\frac{\partial}{\partial t}z(t,x) \in a(t,x)\frac{\partial^2}{\partial x^2}z(t,x) + m(t)b(t, z(t-\sigma(z(t,0))), x), \tag{5.1}$$
$$x \in [0,\pi],\ t \in [0,b], t \neq t_k,$$
$$z(t_k^+, x) - z(t_k^-, x) \in [-b_k|z(t_k^-, x), b_k|z(t_k^-, x)],\quad x \in [0,\pi],\ k = 1,\dots,m, \tag{5.2}$$
$$z(t,0) = z(t,\pi),\quad t \in J := [0,b], \tag{5.3}$$
$$z(t,x) = \phi(t,x),\quad -\infty < t \leq 0,\ x \in [0,\pi], \tag{5.4}$$

where $a(t,x)$ is continuous function and uniformly Hölder continuous in t, $b_k > 0$, $k = 1,\dots,m$, $\phi \in \mathcal{D}$,
$\mathcal{D} = \{\overline{\psi} : (-\infty,0]\times[0,\pi] \to \mathbb{R};\ \overline{\psi}$ is continuous everywhere except for a countable number of points at which $\overline{\psi}(s^-), \overline{\psi}(s^+)$ exist with $\overline{\psi}(s^-) = \overline{\psi}(s)\}$,
$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, $z(t_k^+) = \lim_{(h,x)\to(0^+,x)} z(t_k + h, x)$, $z(t_k^-) = \lim_{(h,x)\to(0^-,x)} z(t_k+h,x)$, $b : \mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cv,k}(\mathbb{R})$ a Carathéodory multivalued map, $\sigma : \mathbb{R} \to \mathbb{R}_+$.

Let

$$y(t)(x) = z(t,x),\quad x \in [0,\pi],\ t \in J = [0,b],$$
$$\mathcal{I}_k(y(t_k^-))(x) = [-b_k|z(t_k^-,x), b_k|z(t_k^-,x)],\quad x \in [0,\pi],\ k = 1,\dots,m,$$
$$F(t,\phi)(x) = b(t)a(t, z(t-\sigma(z(t,0))), x)$$
$$\phi(\theta)(x) = \phi(\theta,x),\quad -\infty < t \leq 0,\ x \in [0,\pi],$$
$$\rho(t,\phi) = t - \sigma(\phi(0,0)).$$

Consider $E = L^2[0,\pi]$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$D(A) = \{w \in E :\ w, w' \text{ are absolutely continuous},\ w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H1) and (H3) (see [17]). For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma = \left\{\phi \in \mathcal{D} : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\right\}$$

with the norm

$$\|\phi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\|\phi(\theta)\|.$$

Notice that the phase space $\mathcal{B}_\gamma$ satisfies axioms (A1) and (A3) (see [25] for more details).

We can show that problem (5.1)–(5.4) is an abstract formulation of problem (1.1)–(1.3). Under suitable conditions, the problem (1.1)–(1.3) has at least one mild solution.

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